Application of perturbation techniques in the analysis of braced frames

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ABSTRACT

The efficiency of applying the incremental perturbation method to combined nonlinear problems was particularly reported by Yokoo, Nakamura and Uetani. (1976) And this successful approach has been extended by Ishida and Morisako (1987) to one dimensional finite element method.

For the sake of studying a macro behaviour of steel frames, this paper deals with the formulation of an ordered set of perturbation equation for a combined nonlinear constitutive equation using strain hardening general yield hinge model to lead to more efficiency on the capacity of computers, a higher speed of calculation and simpler procedures in modeling.

Some examples of braced frames are analysed by the present method, and good correspondence between the results of test and those of analysis on post buckling load - displacement relationship is successfully achieved.

INTRODUCTION

While material and geometrical nonlinear analysis for steel skeleton structures has been done by several investigators, using a series of linear approximations within small increments, the efficiency of applying the incremental perturbation method to combined nonlinear problems was particularly reported by Yokoo, Nakamura and Uetani.(1976) This successful approach has been extended by Ishida and Morisako (1987) to one dimensional finite element method.

However, strain hardening general yield hinge method (Inoue and Ogawa 1978) proves to be more useful than one dimensional finite element method in studying a macro behaviour of frames in terms of the capacity of computers, the speed of calculation and the simple procedures in modeling.

Thus this paper deals with the formulation of an ordered set of perturbation equation for a combined nonlinear constitutive equation using strain hardening general yield hinge model, and some examples of analysis are shown.

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Primary Conditions

The primary conditions for the present analysis are as follows.

1) As a deflection function for each member, a linear expression for the longitudinal direction and the third order polynomial expression for transverse direction are assumed.
2) Yielding can occur only at the ends of the member, and the behaviour during plastic deformation will be characterized by the kinematic hardening rule by Prager. (1955)

In this paper , the m-th coefficient of Taylor series of any variable x will be denoted by $\overset{(m)}{x}$. So x can be expanded about a parameter t as follows.

$$x = \overset{(0)}{x} + \overset{(1)}{x}\, t + \overset{(2)}{x}\, t^2 + \cdots\cdots\cdots \qquad (1)$$

Hence the following relation is given by differentiation.

$$\left.\frac{d^m x}{d t^m}\right|_{t=0} = m!\ \overset{(m)}{x} \qquad (2)$$

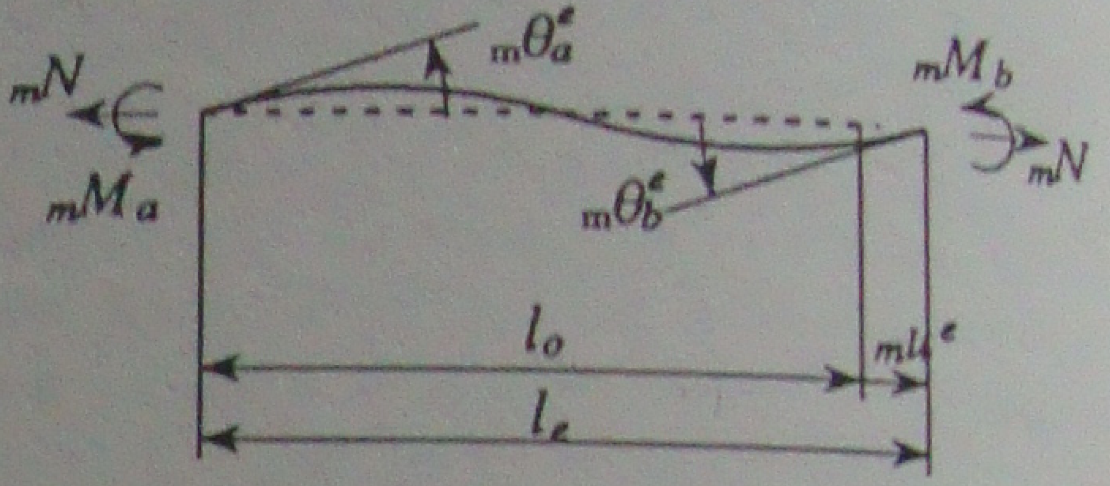

Figure 1

Elastic Constitutive Equation

The elastic relation of member-end forces and member-end elastic deformations are provided by following equations, considering the effect of shortening by bending. (Jennings 1963)

$$_mN = EA\left[\frac{_mu^e}{l_e} + \frac{1}{30}\left\{2\left(_m\theta_a^e\right)^2 - {_m\theta_a^e}\,{_m\theta_b^e} + 2\left(_m\theta_b^e\right)^2\right\}\right] \qquad (3a)$$

$$\begin{aligned}_mM_a = &\left[\frac{4EI}{l_e} + \frac{2EA}{15}\,{_mu^e} + \frac{EA\,l_e}{225}\left\{2\left(_m\theta_a^e\right)^2 - {_m\theta_a^e}\,{_m\theta_b^e} + 2\left(_m\theta_b^e\right)^2\right\}\right]{_m\theta_a^e}\\ &+\left[\frac{2EI}{l_e} - \frac{EA}{30}\,{_mu^e} - \frac{EA\,l_e}{900}\left\{2\left(_m\theta_a^e\right)^2 - {_m\theta_a^e}\,{_m\theta_b^e} + 2\left(_m\theta_b^e\right)^2\right\}\right]{_m\theta_b^e}\end{aligned} \qquad (3b)$$

$$\begin{aligned}_mM_b = &\left[\frac{2EI}{l_e} - \frac{EA}{30}\,{_mu^e} - \frac{EA\,l_e}{900}\left\{2\left(_m\theta_a^e\right)^2 - {_m\theta_a^e}\,{_m\theta_b^e} + 2\left(_m\theta_b^e\right)^2\right\}\right]{_m\theta_a^e}\\ &+\left[\frac{4EI}{l_e} + \frac{2EA}{15}\,{_mu^e} + \frac{EA\,l_e}{225}\left\{2\left(_m\theta_a^e\right)^2 - {_m\theta_a^e}\,{_m\theta_b^e} + 2\left(_m\theta_b^e\right)^2\right\}\right]{_m\theta_b^e}\end{aligned} \qquad (3c)$$

Where superscript e denotes the elastic component of the deformation. Let $_1d^e$ and $_1p$ define by the following equations.

$$_1d^e = \left\{{_mu^e},\ {_m\theta_a^e},\ {_m\theta_b^e}\right\}^T \qquad _1p = \left\{{_mN},\ {_mM_a},\ {_mM_b}\right\}^T \qquad (4),\ (5)$$

Let the right term of Eqs.3 be function f_i , rewriting them as follows.

$$_1p_i = f_i\left({_mu^e},\ {_m\theta_a^e},\ {_m\theta_b^e}\right) \quad ,\ (i = 1, 2, 3) \qquad (6)$$

With Taylor expanding of Eq.6 about a parameter t which denotes the progress of time, and with comparison among the same ordered terms up to the third order, the following equations are obtained.

$$_1\overset{(0)}{p}_i = \overset{(0)}{f}_i \qquad _1\overset{(1)}{p}_i = \sum_{j=1}^{3} \frac{\partial f_i}{\partial\, {_1d_j^e}}\, {_1\overset{(1)}{d}{}_j^e} \qquad (7a),\ (7b)$$

$$\overset{(2)}{{}_1p}_i=\sum_{j=1}^{3}\frac{\partial f_i}{\partial_1 d_j^e}\,\overset{(2)}{{}_1d}{}_i^e+\frac{1}{2!}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2 f_i}{\partial_1 d_j^e\,\partial_1 d_k^e}\,\overset{(1)}{{}_1d}{}_i^e\,\overset{(1)}{{}_1d}{}_k^e \qquad \text{(7c)}$$

$$\overset{(3)}{{}_1p}_i=\sum_{j=1}^{3}\frac{\partial f_i}{\partial_1 d_j^e}\,\overset{(3)}{{}_1d}{}_j^e+\frac{1}{2!}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2 f_i}{\partial_1 d_j^e\,\partial_1 d_k^e}\left(\overset{(1)}{{}_1d}{}_j^e\,\overset{(2)}{{}_1d}{}_k^e+\overset{(2)}{{}_1d}{}_j^e\,\overset{(1)}{{}_1d}{}_k^e\right)$$
$$+\frac{1}{3!}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\partial^3 f_i}{\partial_1 d_j^e\,\partial_1 d_k^e\,\partial_1 d_l^e}\,\overset{(1)}{{}_1d}{}_j^e\,\overset{(1)}{{}_1d}{}_k^e\,\overset{(1)}{{}_1d}{}_l^e \qquad \text{(7d)}$$

It is noted that Eq.7a expresses the equilibrium condition at $t = 0$. Matrix expression of Eqs.7 provides the following ordered set of perturbation equation about the elastic constitutive equation;

$$\overset{(m)}{{}_1p}=K^e\,\overset{(m)}{{}_1d}{}^e+\overset{(m)*1}{{}_1p} \qquad \text{(8)}$$

where

$$\overset{(m)*1}{{}_1p}=\sum_{n=1}^{m-1}\frac{m-n}{(n+1)!}\,{}_2^nS\,\overset{(m-n)}{{}_1d}{}^e \quad (m\geq 2) \qquad \text{(9)}$$

and K^e , ${}_2{}^1S$, ${}_2{}^2S$ are the matrices whose ij component is defined as follows.

$$K_{ij}^e=\frac{\partial f_i}{\partial\,{}_1d_j^e} \qquad {}_2^1S_{ij}=\left\{\frac{\partial^2 f_i}{\partial\,{}_1d_j^e\,\partial\,{}_1d_1^e}\quad\frac{\partial^2 f_i}{\partial\,{}_1d_j^e\,\partial\,{}_1d_2^e}\quad\frac{\partial^2 f_i}{\partial\,{}_1d_j^e\,\partial\,{}_1d_3^e}\right\}\overset{(1)}{{}_1d}{}^e \qquad \text{(10a) , (10b)}$$

$${}_2^2S_{ij}=\left\{{}_2^2S_{ij1}\quad{}_2^2S_{ij2}\quad{}_2^2S_{ij3}\right\}\overset{(1)}{{}_1d}{}^e$$
$${}_2^2S_{ijk}=\left\{\frac{\partial^3 f_i}{\partial_1 d_j^e\,\partial_1 d_k^e\,\partial_1 d_1^e}\ \cdots\ \frac{\partial^3 f_i}{\partial_1 d_j^e\,\partial_1 d_k^e\,\partial_1 d_3^e}\right\}\overset{(1)}{{}_1d}{}^e \qquad \text{(10c)}$$

Plastic Constitutive Equation

Now we consider the case when yield hinges form at either end of the member. Let the member-end deformation, ${}_1d = \{{}_mu\ ,\ {}_m\theta_a\ ,\ {}_m\theta_b\}^T$, and member-end force be divided into components at each end, a and b. And we define as follows;

$${}_1d_{(a)}=\{\ {}_mu_a\quad {}_m\theta_a\ \}^T \qquad {}_1d_{(b)}=\{\ {}_mu_b\quad {}_m\theta_b\ \}^T \qquad \text{(11a) , (11b)}$$

$${}_1p_{(a)}=\{\ {}_mN\quad {}_mM_a\ \}^T \qquad {}_1p_{(b)}=\{\ {}_mN\quad {}_mM_b\ \}^T \qquad \text{(12a) , (12b)}$$

where

$${}_mu={}_mu_a+{}_mu_b \qquad \text{(13)}$$

According to Prager (1955), if the yield surface is drawn with the axes of the stresses divided by yield stresses and those of plastic strains multiplied by yield stresses, the subsequent yield surface will move parallel in the direction of plastic strain without changing its shape and scale as Fig.2. Yield surfaces are defined at both a and b ends of the member, and following equation is provided when yielding occurs at both ends.

$$F\left({}_1p_{(a,b)}-\alpha_{(a,b)}\right)=0 \qquad \text{(14)}$$

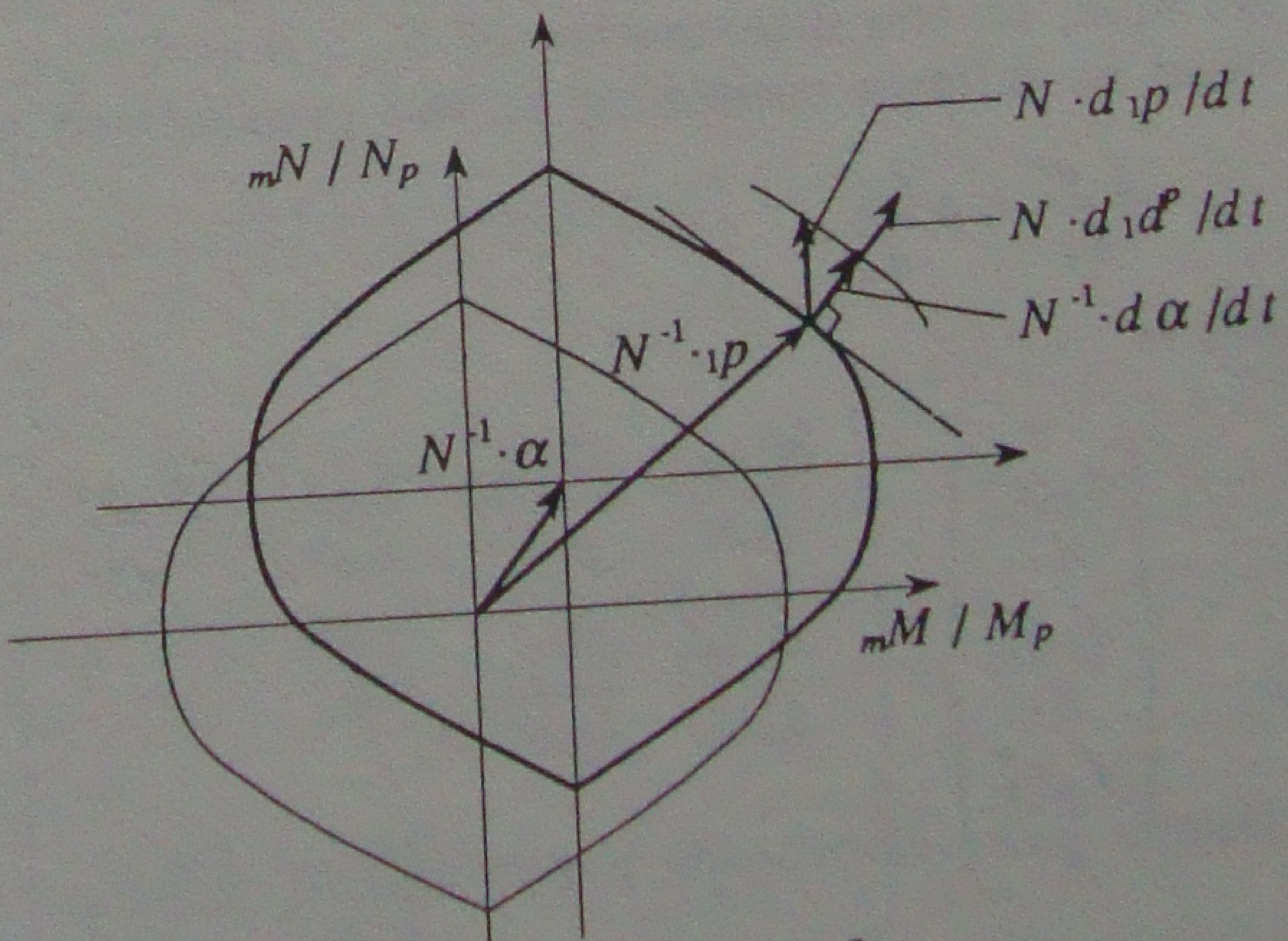

Figure 2

The sign (a,b) in Eq.14 expresses that similar equations are provided for both a and b ends. $\alpha_{(a)}$ and $\alpha_{(b)}$ express the movement vector of the origin of the yield surface respectively at a end and b end and according to Prager's kinematic hardening rule they are related to plastic deformation velocity ${}_1d^p/dt$ with scalar amount μ, that is;

$$\frac{d\,\alpha_{(a,b)}}{dt} = \mu_{(a,b)} N N \frac{d\,{}_1d^p_{(a,b)}}{dt} \tag{15}$$

where N is the diagonal matrix with the components of yield stresses N_p and M_p. The yield condition that the stress point remains on the subsequent yield surface provides the following equation.

$$\left\{\frac{d\,{}_1d^p_{(a,b)}}{dt}\right\}^T \left(\frac{d\,{}_1p_{(a,b)}}{dt} - \frac{d\,\alpha_{(a,b)}}{dt}\right) = 0 \tag{16}$$

Taylor expansion of Eq.15 provides an ordered set of perturbation equation as follows;

$$\overset{(m)}{\alpha}_{(a,b)} = \frac{1}{m} N N \overset{(m-1)}{\mu}_{(a,b)}\, {}_1\overset{(1)}{d}{}^p_{(a,b)} + \overset{(m)*1}{\alpha}_{(a,b)} \tag{17}$$

where

$$\overset{(m)*1}{\alpha}_{(a,b)} = N N \sum_{n=2}^{m} \frac{n}{m} \overset{(m-n)}{\mu}_{(a,b)}\, {}_1\overset{(n)}{d}{}^p_{(a,b)} \qquad (m \geq 2) \tag{18}$$

Similarly, Eq.16 may be expanded to Taylor series and the substitution of Eq.17 provides the following ordered set of equation about μ ;

$$\overset{(m-1)}{\mu}_{(a,b)} = \left({}_1\overset{(1)}{d}{}^{p\,T}_{(a,b)} N N\, {}_1\overset{(1)}{d}{}^p_{(a,b)}\right)^{-1} m\, {}_1\overset{(1)}{d}{}^{p\,T}_{(a,b)}\, {}_1\overset{(m)}{p}_{(a,b)} + \overset{(m-1)*1}{\mu}_{(a,b)} \tag{19}$$

where

$$\overset{(m-1)*1}{\mu}_{(a,b)} = \left({}_1\overset{(1)}{d}{}^{p\,T}_{(a,b)} N N\, {}_1\overset{(1)}{d}{}^p_{(a,b)}\right)^{-1} \left\{ -m\, {}_1\overset{(1)}{d}{}^{p\,T}_{(a,b)} \overset{(m)*1}{\alpha}_{(a,b)} + \sum_{n=2}^{m} n(m-n+1)\, {}_1\overset{(n)}{d}{}^{p\,T}_{(a,b)} \left({}_1\overset{(m-n+1)}{p}_{(a,b)} - \overset{(m-n+1)}{\alpha}_{(a,b)}\right)\right\} \qquad (m \geq 2) \tag{20}$$

The velocity of the member-end deformation $d_1 d/dt$ is given as a summation of the elastic component $d_1 d^e/dt$ and the plastic component $d_1 d^p/dt$, that is

$$\frac{d\,{}_1d}{dt} = \frac{d\,{}_1d^e}{dt} + \frac{d\,{}_1d^p}{dt} \tag{21}$$

If the stress points at both a and b ends are on the singular point of the yield surface (for example, on the intersection of $F_1 = 0$ and $F_2 = 0$), the velocity of plastic deformation is provided as follows according to the generalized plastic flow rule;

$$\frac{d\,{}_1d^p}{dt} = \phi \frac{d\,\zeta}{dt} \tag{22a}$$

that is

$$\begin{Bmatrix} \dfrac{d\,{}_m u^p}{dt} \\ \dfrac{d\,{}_m\theta^p_a}{dt} \\ \dfrac{d\,{}_m\theta^p_b}{dt} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial F_{1(a)}}{\partial_m N} & \dfrac{\partial F_{2(a)}}{\partial_m N} & \dfrac{\partial F_{1(b)}}{\partial_m N} & \dfrac{\partial F_{2(b)}}{\partial_m N} \\ \dfrac{\partial F_{1(a)}}{\partial_m M_a} & \dfrac{\partial F_{2(a)}}{\partial_m M_a} & 0 & 0 \\ 0 & 0 & \dfrac{\partial F_{1(b)}}{\partial_m M_b} & \dfrac{\partial F_{2(b)}}{\partial_m M_b} \end{bmatrix} \begin{Bmatrix} \dfrac{d\,\zeta_{1(a)}}{dt} \\ \dfrac{d\,\zeta_{2(a)}}{dt} \\ \dfrac{d\,\zeta_{1(b)}}{dt} \\ \dfrac{d\,\zeta_{2(b)}}{dt} \end{Bmatrix} \tag{22b}$$

When the yield hinge formation is different from that in the above case, $d_1 d^p/dt$ may be provided by adequate squeezing of ϕ and $d\,\zeta/dt$.

Let $d\alpha_o/dt$ be defined by the following equation as a product of the elastic stiffness matrix K^e, the deformation velocity $d_1 d/dt$ and strain hardening coefficient τ.

$$\frac{d\,\alpha_o}{dt} = \tau K^e \frac{d_1 d}{dt} \tag{23}$$

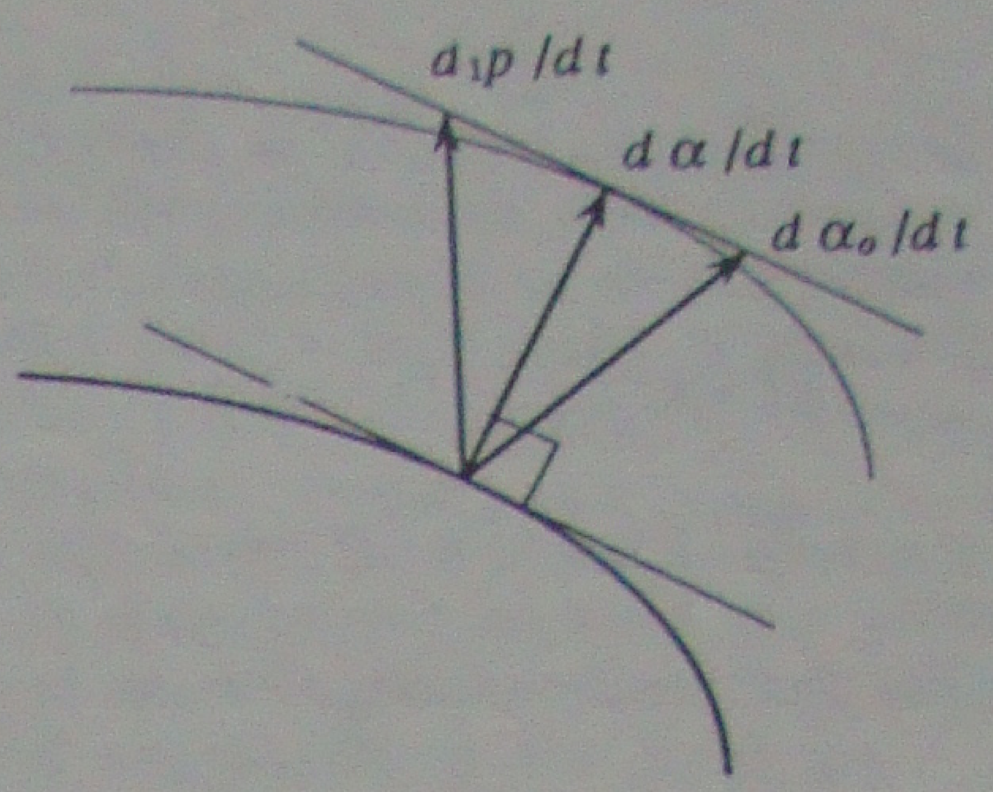

Figure 3

Although $d\alpha_o/dt$ is generally unequal to $d\alpha/dt$ except in the case of uniaxial loading, let $d\alpha/dt$ approximate to the normal component of $d\alpha_o/dt$ (normal to the yield surface). In this case, the yield condition that the stress point remains on the subsequent yield surface may be written as follows. (refer to Fig.3)

$$\phi^T\left(\frac{d_1 p}{dt} - \frac{d\,\alpha_o}{dt}\right) = 0 \tag{24}$$

Taylor expansion of Eq.22 provides the following ordered set of equation;

$${}_1\overset{(m)}{d}{}^p = \overset{(0)}{\phi}\,\overset{(m)}{\zeta} + {}_1\overset{(m)*1}{d}{}^p \tag{25}$$

where

$${}_1\overset{(m)*1}{d}{}^p = \sum_{n=1}^{m-1}\frac{n}{m}\,\overset{(m-n)}{\phi}\,\overset{(n)}{\zeta} \tag{26}$$

Similarly, from Eq.21 the following equation is provided.

$${}_1\overset{(m)}{d}{}^e = {}_1\overset{(m)}{d} - {}_1\overset{(m)}{d}{}^p \tag{27}$$

By substituting Eqs.25 and 27 for Eq.8 , ${}_1\overset{(m)}{p}$ is expressed as follows.

$${}_1\overset{(m)}{p} = K^e\left({}_1\overset{(m)}{d} - \overset{(0)}{\phi}\,\overset{(m)}{\zeta}\right) - K^e\,{}_1\overset{(m)*1}{d}{}^p + {}_1\overset{(m)*1}{p} \tag{28}$$

From Eqs.23 and 24, the following equations are respectively provided.

$$\overset{(m)}{\alpha}_o = \tau K^e\,{}_1\overset{(m)}{d} \tag{29}$$

$$m\,\overset{(0)}{\phi}{}^T\,{}_1\overset{(m)}{p} + \sum_{n=1}^{m-1} n\,\overset{(m-n)}{\phi}{}^T\,{}_1\overset{(n)}{p} = m\,\overset{(0)}{\phi}{}^T\,\overset{(m)}{\alpha}_o + \sum_{n=1}^{m-1} n\,\overset{(m-n)}{\phi}{}^T\,\overset{(n)}{\alpha}_o \tag{30}$$

Substituting Eqs.28 and 29 for Eq.30 provides the following equation;

$$\overset{(m)}{\zeta} = (1-\tau)\,C^{-1}\,\overset{(0)}{\phi}{}^T K^e\,{}_1\overset{(m)}{d} + \overset{(m)*1}{\zeta} \tag{31}$$

where

$$\overset{(m)*1}{\zeta} = C^{-1}\left\{-\overset{(0)}{\phi}{}^T K^e\,{}_1\overset{(m)*1}{d}{}^p + \overset{(0)}{\phi}{}^T\,{}_1\overset{(m)*1}{p} + \sum_{n=1}^{m-1}\frac{n}{m}\,\overset{(m-n)}{\phi}{}^T\left({}_1\overset{(n)}{p} - \overset{(n)}{\alpha}_o\right)\right\} \qquad (m>2) \tag{32}$$

$$C = \overset{(0)}{\phi}{}^T K^e\,\overset{(0)}{\phi} \tag{33}$$

By substituting Eq.31 for Eq.28, we can obtain the ordered set of the constitutive equation under loading as follows;

$$ {}_1\overset{(m)}{p} = K^p \, {}_1\overset{(m)}{d} + {}_1\overset{(m)*2}{p} \tag{34}$$

where

$$ K^p = K^e \left\{ I - (1-\tau)\, \overset{(0)}{\phi}\, C^{-1}\, \overset{(0)}{\phi}{}^T K^e \right\} \tag{35}$$

$$ {}_1\overset{(m)*2}{p} = - K^e \left\{ \overset{(0)}{\phi}\, \overset{(m)*1}{\zeta} + {}_1\overset{(m)*1}{d}{}^p \right\} + {}_1\overset{(m)*1}{p} \tag{36}$$

I : unit matrix

EXAMPLES OF ANALYSIS

The frames considered as examples are the two-story braced frames as shown in Fig.4 which are the specimens for a static alternate loading test we conducted. (Igarashi, 1986) A node is set at the mid-point of each bracing member which is rigidly fixed at ends to the frame. Initial deflection of each bracing member e is set to have the amount of l /500 in the lateral direction at the mid-point, where l is the length of the bracing member. The characteristics of the member, either at the panel zone or at the region where the gusset plate is connected, are considered as the stiffness being double and the strength being infinite. The

Table 1

Member	Section	A (cm^2) / Np (ton)	I (cm^4) / Mp (t cm)
Beam	H - 125 x 125 x 6.5 x 9	27.2	774
		85.7	442
Column	H - 150 x 150 x 7 x 10	36.6	1510
		108	668
Bracing	WH - 80 x 80 x 6 x 6	13.2	49.3
		52.5	58.3

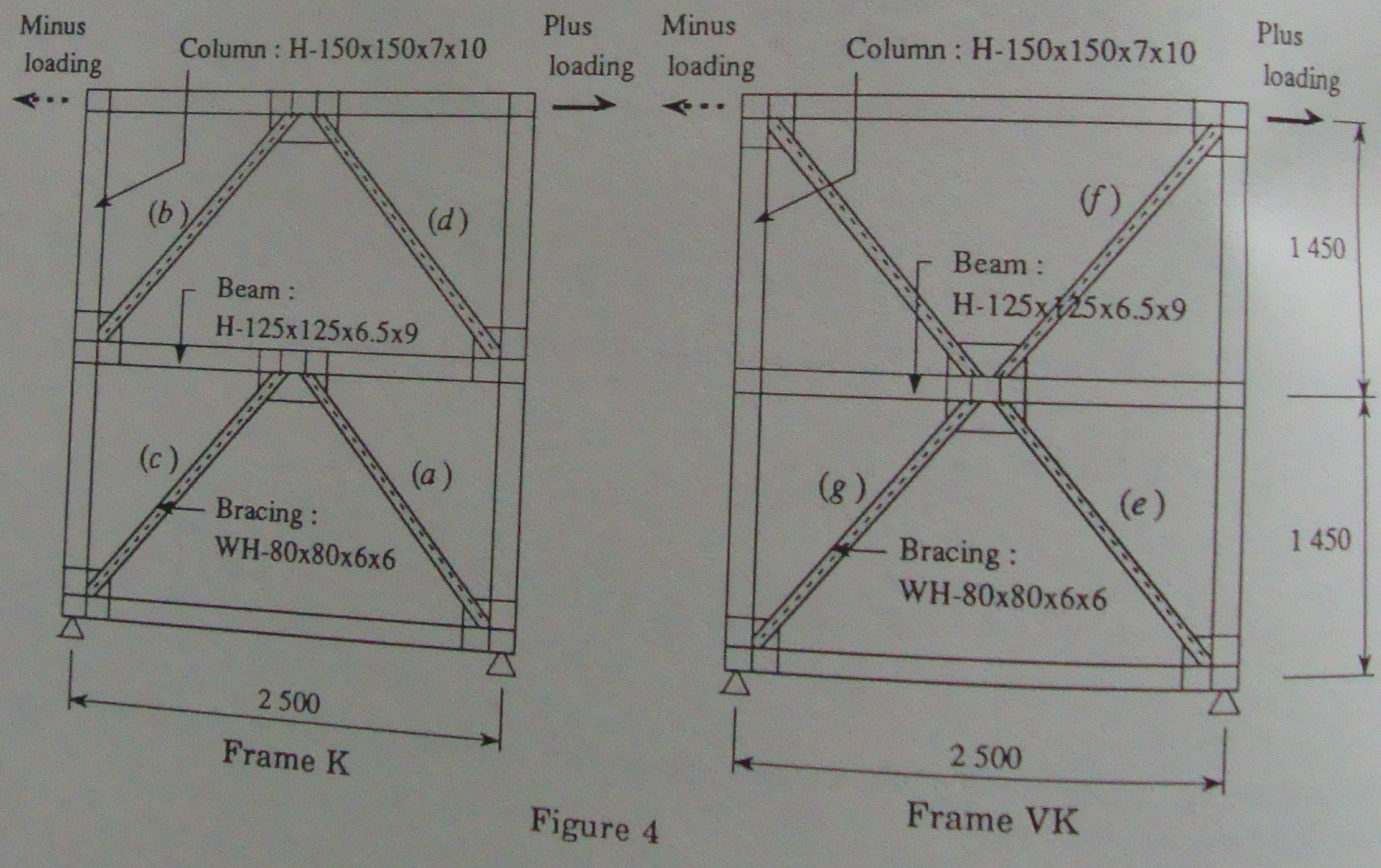

Figure 4

parabolic yield functions are assumed for all members.

Ordered set of perturbation equations up to the third order are considered in the present analysis. The length of increment for step-by-step analysis is determined so that the tolerance limit (Yokoo, 1976) be 0.01 for the horizontal and vertical displacements at the joints, among the column, the beam and the bracing members, and at the mid-point of bracing members. Yielding and unloading of the member and the stress point reaching to the singular point during loading also determine the length of increment.

General stiffness equation is solved by controlling the displacement largest in the previous increment. The direction of controlled displacement is determined so that the yield hinge occurring at the last do not unload. The resulting post buckling behaviour that both load and displacement decrease is successfully achieved and expressed in the analysis.

The results of test and present analysis on the relationship between load and the average relative story rotation are shown in Fig.5. Although the analytical result, compared with the test result, has sharp peaks around the buckling points owing to the hinge method, the relative story rotation and the load level when buckling occurs correspond well with the test result. (a) - (g) in Fig.5 shows the occurrence of buckling. Each buckling member corresponds to the bracing member denoted by (a) - (g) in Fig.4. All buckled members in the analysis correspond with those buckled in the test for the frame K. But, for the frame VK, bracing member (g) has buckled at the first minus-loading in the analysis, while the bracing member (f) has buckled in the test. It is considered that because the stress levels of both (f) and (g) members are nearly equal, the slightest effect on the stress level caused by some deviations from the test specimens and the mathematical models can easily change the collapse mechanism. In fact, the stress level of the member (f), when the member (g) was buckled in the analysis, was $(N/Np)^2 + (M/Mp) = 0.987$, where 1.0 is the yielding stress level. Although the collapse mechanisms are different, the analytical result corresponds well with the test result on the the load - average relative story rotation relationship afterwards in this example.

The load - average relative story rotation relationship for the frame K, analysed under the condition that the initial deflection e is equal to zero, is shown in Fig.6. The initial buckling load rises about 13% more compared to the case of $e = l/500$. The bracing members (c) and (d) have not buckled. The reason for the member (c) is that the right end of the member (c) yielded at the point of sign △ in the figure because of the rising of the initial buckling load and it experienced the plastic deformation in the direction opposite to the buckling mode.

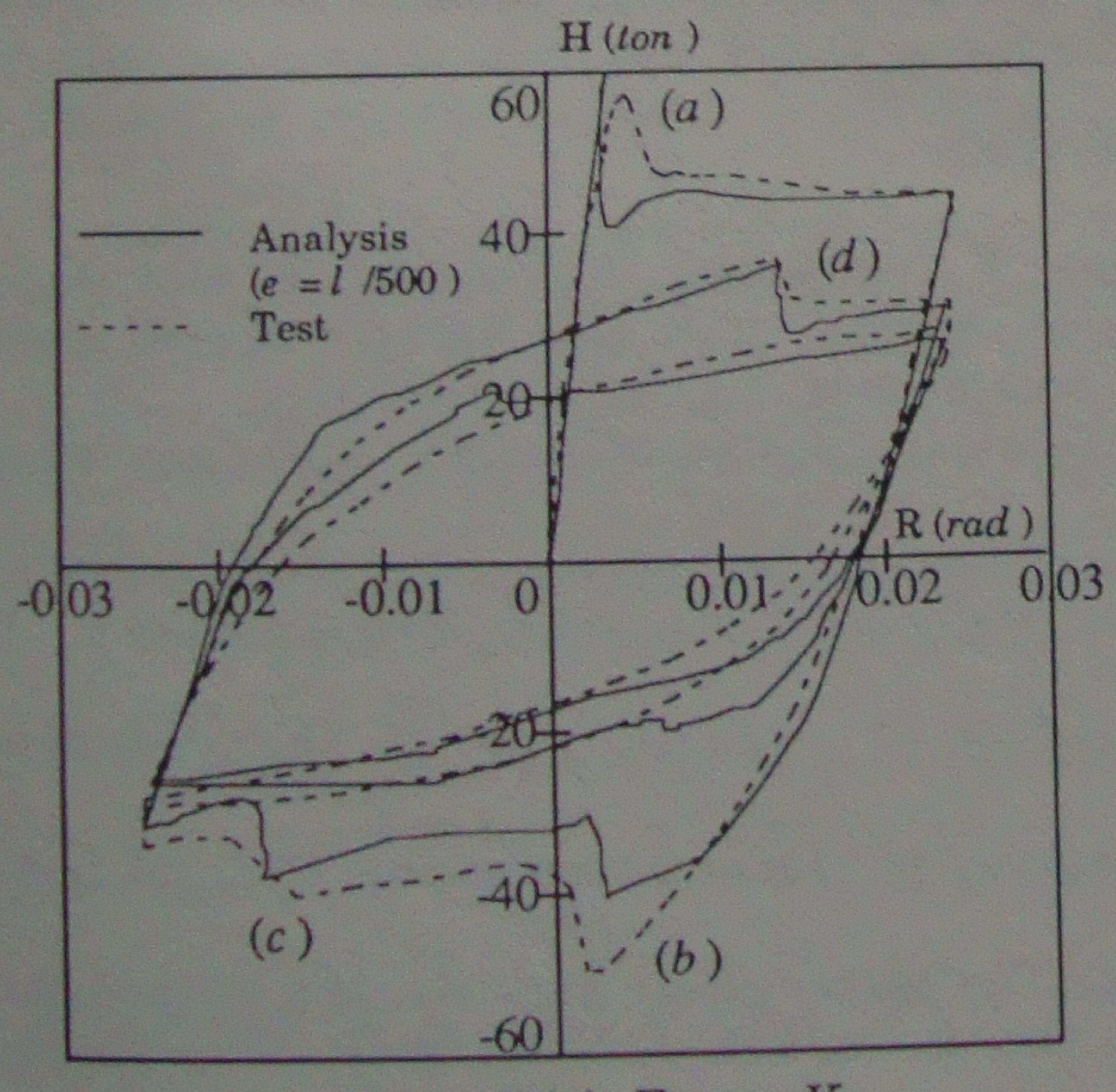

Figure 5 (a) Frame K

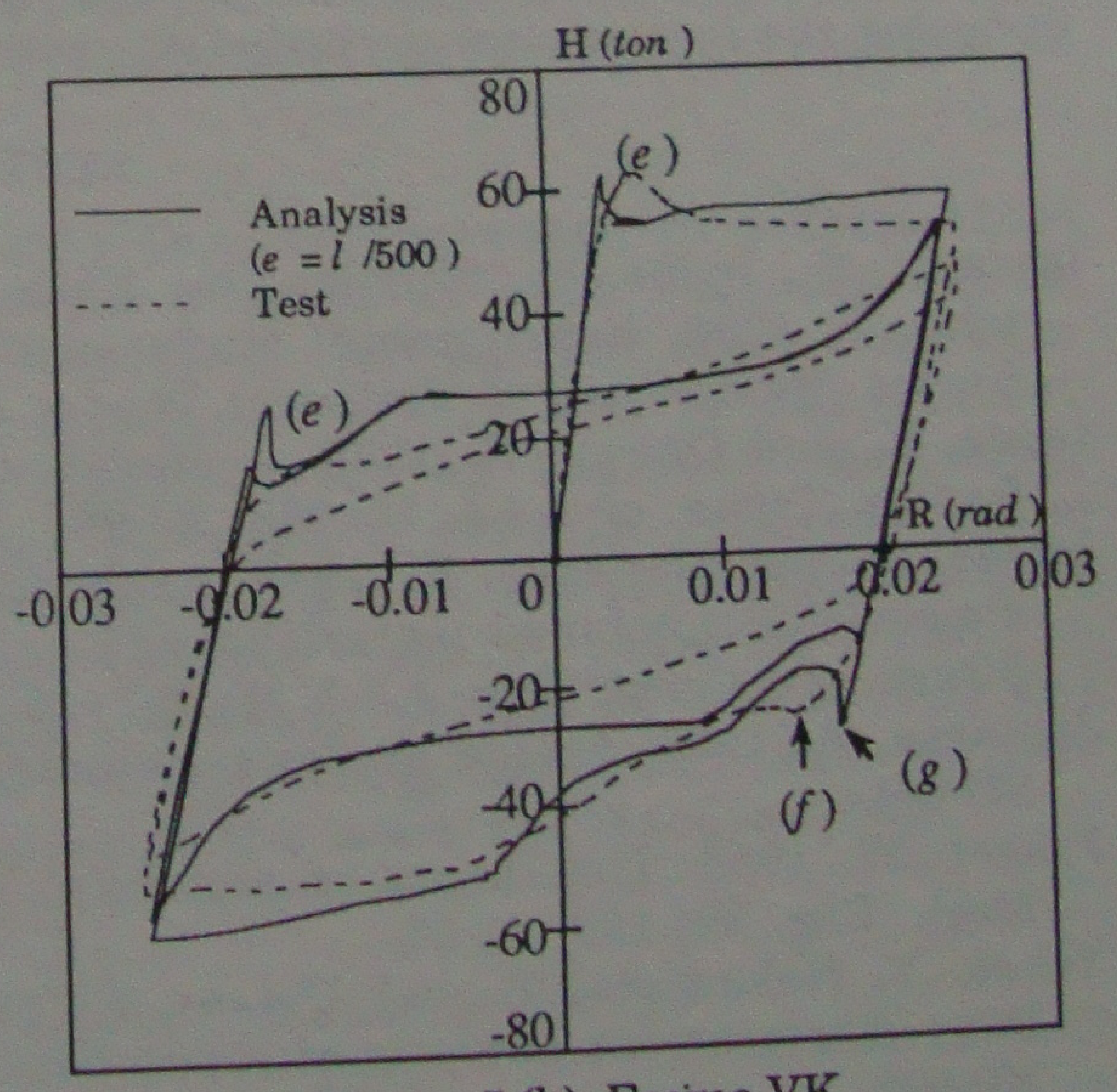

Figure 5 (b) Frame VK

The reason for the member (d) is considered as follows. The upper end of the right hand column at the second story yielded at the point of sign ▲ and experienced great plastic deformation when $e = 0$ because of the member (c) not buckling, while the section was immediately unloaded when $e = l/500$. This caused the change of the moment distribution at the following plus-loading phase and obstructed the buckling of the member (d). Thus, the difference in the initial deflection amounts of the member causes not only the load level of the initial buckling but also the ensuing load - displacement relationship.

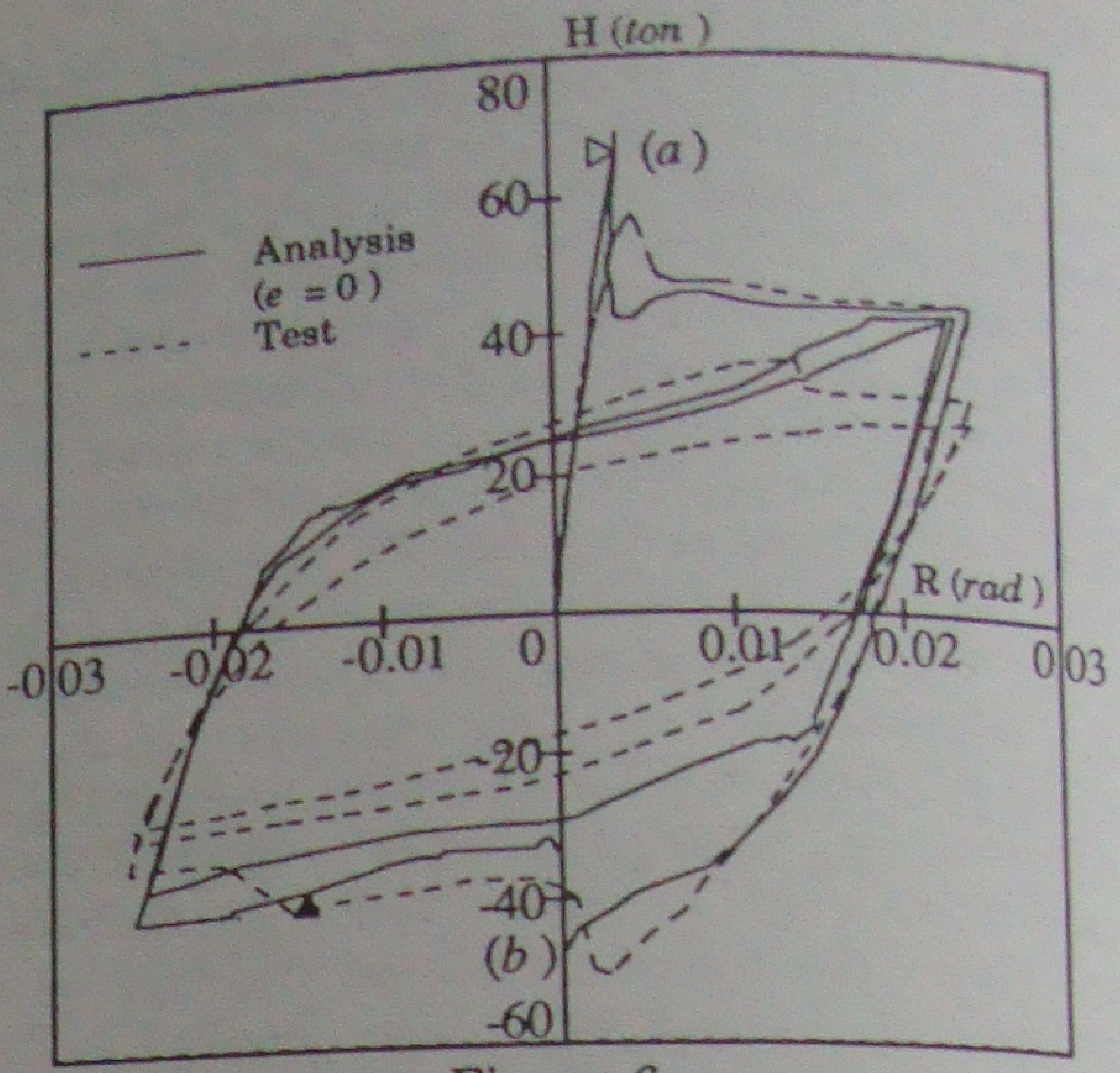

Figure 6

CONCLUSION

Conclusions obtained are summarized as follows.

1. The ordered set of perturbation equation of combined nonlinear constitutive equation can be obtained for the strain hardening general yield hinge model.
2. The present method of analysis gives good estimation for the load - displacement relationship of the braced skeleton when it is accompanied by buckling.
3. The load - displacement relationship analysed by general yield hinge method has sharp peaks around the buckling points because the model cannot express the gradual expansion of the plastic region.
4. The initial deflection of the member effects the initial buckling load level and can also effect the load - displacement relationship afterwards.

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